

QUARTERLY PROGRESS REPORT

**Study of Recirculating Gas Flow Fields in
the Base Region of Saturn-Class Vehicles**

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Abstract

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The mathematical correspondence between low speed and high speed turbulent boundary layers which was developed by Coles⁽¹⁾ and extended by Crocco⁽²⁾ is extended to multi-specie and chemically reacting flows. It is found that the Prandtl and Lewis number transformations are coupled, except in the singular case where all such numbers are unity. A simple boundary layer situation is examined to determine the quantitative nature of the coupling. It is concluded that the Coles-Crocco transformation scheme generally requires physically unrealistic values of transport properties when chemistry is considered.

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1. Introduction

This report covers the period August 1 - October 31, 1965. Investigations accomplished in the reporting period included studies of 1) the transformation of a chemically reacting high speed turbulent boundary layer to an equivalent low speed flow, 2) determination of the most important reactions in the free shear-afterburning layer and 3) studies of the effect of turbulent mixing delay on overall reaction kinetics. Only study (1) is reported on in detail here. A succeeding report, to be produced in the immediate future, will contain detailed information on studies (2) and (3). The work reported herein was accomplished primarily by Mr. H. F. Nelson.

2. Nomenclature

- ϵ turbulent viscosity
 μ laminar viscosity
 ρ, P, T density, pressure, temperature
 u, v velocity in x, y direction
 τ shear stress = $(\mu + \epsilon) \partial u / \partial y$
 H total enthalpy = $h + \frac{1}{2} u^2$
 h = $\sum Y_i h_i$
 h_i = $\int_0^T c_{p_i} dT + \Delta h_i^\circ$
 c_p^* = $\sum Y_i c_{p_i}$
 $\dot{\omega}_i$ = rate of formation of species i
 Δh_i° heat of formation
 q_c conduction heat transfer = $\frac{\mu + \epsilon}{P_e} \frac{\partial h}{\partial y}$
 q_D diffusion heat transfer = $\frac{\mu + \epsilon}{P_e} (L_e - 1) \sum_i h_i \frac{\partial Y_i}{\partial y}$
 β_1 = $[\mu + \epsilon] / s_e \frac{\partial Y_1}{\partial y}$
 Y_i = mass fraction of species i
 ψ stream function
 δ boundary layer thickness
 δ^* displacement thickness
 θ momentum thickness
 c_f coefficient of friction $\tau_w = \frac{1}{2} \rho_e u^2 c_f$
 ϕ enthalpy thickness

γ_i specie i thickness

Q heat transfer = $q_c + u\tau + q_D$

C_H heat transfer coefficient

C_{HD} diffusion heat transfer coefficient

$\sigma, \eta, \xi, \alpha, \lambda_1$ transformation variables

subscripts

e edge of boundary layer except P_e, L_e, S_e which is effective
 Pr, L, Sc (see eqs.1)

r recovery value

w wall

superscript

$-$ low speed value

3. Transformation of Chemically Reacting Turbulent Boundary Layers

The work described in this section is an investigation of the extension to reacting flows of the transformation of the turbulent boundary layer suggested by Coles⁽¹⁾ and extended by Crocco⁽²⁾. This transformation technique has been applied to reacting mixing layers by Casaccio⁽³⁾ and to reaction boundary layers by Rosenbaum⁽⁴⁾. However, neither of the latter authors considered the mathematical constraints on transport processes imposed by extension of the transformation technique to the species diffusion equation. The emphasis here is thus placed on finding a transformation between the low speed and high speed Prandtl and Lewis numbers.

In the works of Casaccio⁽³⁾ and Rosenbaum⁽⁴⁾ the coupling of Lewis and Prandtl numbers in the two flow regimes was completely neglected since $Pr = Pr_t = L = L_t = 1$. In the work of Crocco⁽²⁾ the coupling of the two Prandtl numbers was considered, but that of Lewis numbers neglected since $L = L_t = 1$. Coles⁽¹⁾ was concerned only with the momentum transformation, i.e., with the proper correspondence of the pressure and inertia terms in both the physical and transformed planes.

The conservation equations describing the mean properties of the hypersonic turbulent reacting two-dimensional boundary layer are taken to be the same as for laminar flow with the molecular transport parameters replaced by their turbulent counterparts. Thus effective Prandtl, Schmidt and Lewis numbers can be defined as⁽⁴⁾

$$\frac{1}{Pe} = \left[\frac{1}{Pr(1+g/\mu)} + \frac{1}{Pr_t(1+\mu_0/\mu)} \right] \quad (1.a)$$

$$\frac{1}{S_e} = \left[\frac{1}{S_e (1 + \epsilon/\mu)} + \frac{1}{S_{e_e} (1 + \mu/\epsilon)} \right] \quad (1.b)$$

$$L_e = P_e / S_e \quad (1.c)$$

Now the hypersonic turbulent, reacting boundary layer equations can be written as

Continuity

$$\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0 \quad (2.a)$$

Momentum - x -

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = - \frac{\partial P}{\partial x} + \frac{\partial \tau}{\partial y} \quad (2.b)$$

Energy

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y} \left[q_c + u \tau + q_D \right] \quad (2.c)$$

Species

$$\rho u \frac{\partial Y_i}{\partial x} + \rho v \frac{\partial Y_i}{\partial y} = \frac{\partial \beta_i}{\partial y} + \dot{W}_i \quad (2.d)$$

where the following definitions have been used

Mass diffusivity

$$\rho D + \rho D_t = \frac{\mu + \epsilon}{S_e} \quad (3.a)$$

Thermal conductivity

$$k = (\mu + \epsilon) \frac{\bar{C}_p}{P_e} \quad (3.b)$$

Shear stress

$$(u+E) \frac{\partial u}{\partial y} = \tau \quad (3.c)$$

Heat transfer

$$\frac{u+E}{P_e} \frac{\partial H}{\partial y} + (u+E) \left(1 - \frac{1}{P_e}\right) u \frac{\partial u}{\partial y} = q_c + u\tau \quad (3.d)$$

Diffusion heat transfer

$$\frac{u+E}{P_e} (L_e - 1) \sum h_i \frac{\partial Y_i}{\partial y} = q_D \quad (3.e)$$

$$\frac{u+E}{S_e} \frac{\partial Y_i}{\partial y} = \beta_i \quad (3.f)$$

Following References (1, 2, and 3) a general transformation of variables is introduced

$$\frac{\bar{\Psi}(\bar{x}, \bar{y})}{\Psi(x, y)} = \sigma(x) \quad (4.a)$$

$$\frac{\bar{P}}{P} \frac{\partial \bar{y}}{\partial y} = \eta(x) \quad (4.b)$$

$$\frac{d\bar{x}}{dx} = \xi(x) \quad (4.c)$$

$$\frac{\bar{Y}_i(\bar{x}, \bar{y})}{Y_i(x, y)} = \lambda_i(x) \quad (4.d)$$

$$\frac{\overline{H}(\overline{x}, \overline{y})}{H(x, y)} = \alpha(x) \quad (4.e)$$

This transformation is intended to transform the hypersonic turbulent reacting boundary layer to a low-speed reacting boundary layer. The barred quantities represent the low speed variables, and $\sigma(x)$, $\eta(x)$, $\xi(x)$, $\alpha(x)$ and $\lambda_1(x)$ are unknown functions of x . The stream function is defined as

$$\begin{aligned} \frac{\partial \overline{\psi}}{\partial \overline{x}} &= -\overline{\rho} \overline{v} & \frac{\partial \psi}{\partial x} &= -\rho v \\ \frac{\partial \overline{\psi}}{\partial \overline{y}} &= \overline{\rho} \overline{u} & \frac{\partial \psi}{\partial y} &= \rho u \end{aligned} \quad (5)$$

It has been shown in References 1, 2, and 3 that

$$\frac{\overline{u}}{u} = \frac{\overline{u_e}}{u_e} = \frac{\sigma}{\eta} \quad (6)$$

$$\frac{\partial \overline{P}}{\partial \overline{x}} = \frac{\overline{\rho_e}}{\rho_e} \frac{\sigma^2}{\eta^2 \xi} \left[\frac{\partial P}{\partial x} + \rho_e u_e^2 \frac{d}{dx} \left[\ln \frac{\eta}{\sigma} \right] \right] \quad (7)$$

It has been shown in reference 2 that for the momentum equations to have the same form in both the high and low speed boundary layers the shear must transform as

$$\begin{aligned} \bar{\tau} = \frac{\sigma^2}{\eta \xi} \left[\tau + \frac{d}{dx} (\ln \sigma) \int_y^\delta \psi \frac{\partial u}{\partial y} dy + \frac{dP}{dx} \int_y^\delta \left(1 - \frac{\bar{p}_e P}{\bar{p}_e \bar{P}} \right) dy \right. \\ \left. + \bar{p}_e \frac{u_e^2}{\sigma} \frac{d}{dx} \left(\ln \frac{\eta}{\sigma} \right) \int_y^\delta \frac{P}{\bar{p}_e} \left(\frac{u^2}{u_e^2} - \frac{\bar{p}_e}{\bar{P}} \right) dy \right] \end{aligned} \quad (8)$$

By defining Newtonian friction at the wall in the same manner as in reference 2, $\xi(x)$ becomes

$$\begin{aligned} \xi(x) = \sigma \eta \frac{\bar{p}_w \mu_w}{\bar{p}_w \bar{\mu}_w} \left(1 + \frac{2}{C_f} \left[\theta \frac{d}{dx} (\ln \sigma) - \frac{1}{u_e} \frac{d u_e}{dx} \int_0^\delta \left(1 - \frac{\bar{p}_e P}{\bar{P} \bar{p}_e} \right) dy \right. \right. \\ \left. \left. - \left[\delta^* + \theta - \int_0^\delta \left(1 - \frac{\bar{p}_e P}{\bar{P} \bar{p}_e} \right) dy \right] \frac{d}{dx} \left(\ln \frac{\eta}{\sigma} \right) \right] \right) \end{aligned} \quad (9)$$

Similarly, in order that the two energy equations be of the same form, the heat transfer must transform as

$$\bar{Q} = \frac{\sigma \alpha}{\xi} \left(q_c + u \bar{\tau} + q_D + \frac{d}{dx} (\ln \sigma) \int_y^\delta \psi \frac{\partial H}{\partial y} dy - \frac{d}{dx} (\ln \sigma) \int_y^\delta \rho u H dy \right) \quad (10)$$

Defining

$$\bar{Q}_w = \frac{\bar{\mu}_w}{\bar{p}_{ew}} \left(\frac{\partial H}{\partial y} \right)_w + (\bar{L}_{ew} - 1) \sum_i (\bar{h}_i \frac{\partial \bar{Y}_i}{\partial y})_w \quad (11)$$

and evaluating equation (10) at the wall

$$\frac{\bar{Q}_w}{Q_w} = \frac{\alpha \bar{p}_{ew} \bar{\mu}_w \bar{P}_w}{\eta \bar{p}_{ew} \mu_w \bar{P}_w} \left[\frac{1 + (\bar{L}_{ew} - 1)(1 - \bar{c}_{Pw}^* \bar{T}_e / \bar{H}_e)}{1 + (\bar{L}_{ew} - 1)(1 - \bar{c}_{Pw}^* T_e / H_e)} \right] \quad (12)$$

where $T = T_e \frac{H}{H_e}$ $\bar{T} = \bar{T}_e \frac{\bar{H}}{\bar{H}_e}$

Defining a heat transfer coefficient C_H and a diffusion heat transfer coefficient C_{HD} so that

$$Q_w = C_H \rho_e u_e (H_r - H_w) + C_{HD} \rho_e u_e \sum_i h_i (Y_{ie} - Y_{iw})$$

Following reference 2 another relation for $\xi(x)$ can be written

$$\xi(x) = \eta \sigma' \frac{\mu_w \rho_w}{\mu_w \rho_w} \left(\frac{1 + (L_{ew} - 1)(1 - C_{pw}^* T_e / H_e)}{1 + (\bar{L}_{ew} - 1)(1 - \bar{C}_{pw}^* \bar{T}_e / \bar{H}_e)} \right) \frac{\bar{P}_{ew}}{P_{ew}} \quad (13)$$

$$\left[1 + \frac{H_e \left[\phi \frac{d}{dx} (\ln \sigma) + (\phi + \delta^* - \delta) \frac{d}{dx} (\ln \alpha) \right]}{C_H (H_r - H_w) + C_{HD} \sum_i h_i (Y_{ie} - Y_{iw})} \right]$$

Combining equations (9) and (13) we get a relationship between σ , α , and η .

$$1 + \frac{2}{C_f} \left(\theta \frac{d}{dx} (\ln \sigma) - \frac{1}{u_e} \frac{du_e}{dx} \int_0^\delta \left(1 - \frac{\bar{P}_e \rho}{\bar{P} \rho_e} \right) dy \right. \\ \left. - \left[\delta^* + \theta - \int_0^\delta \left(1 - \frac{\bar{P}_e \rho}{\bar{P} \rho_e} \right) dy \right] \frac{d}{dx} (\ln \eta / \sigma) \right) = \quad (14)$$

$$\frac{\bar{P}_{ew}}{P_{ew}} \left[1 + \frac{H_e \left[\phi \frac{d}{dx} (\ln \sigma) + (\phi + \delta^* - \delta) \frac{d}{dx} (\ln \alpha) \right]}{C_H (H_r - H_w) + C_{HD} \sum_i h_i (Y_{ie} - Y_{iw})} \right] \left[\frac{1 + (L_{ew} - 1)(1 - C_{pw}^* T_e / H_e)}{1 + (\bar{L}_{ew} - 1)(1 - \bar{C}_{pw}^* \bar{T}_e / \bar{H}_e)} \right]$$

Equation (14) reduces to equation 2.25 of reference 2 for the case

$\bar{P}_{ew} = P_{ew}$, $L_{ew} = \bar{L}_{ew} = 1$, $C_{HD} = 0$, and $\alpha = \text{constant}$. Note that if the Lewis numbers are different from

unity, the reduction cannot be accomplished. Solving equation (14)

for $\ln \alpha$ we have

$$\begin{aligned} \ln \alpha = & \int_0^{\delta} \left[\left(1 + \frac{2}{C_f} \left\{ \sigma \frac{d}{dx} (\ln \sigma) - \frac{1}{u_e} \frac{du_e}{dx} \int_0^{\delta} \left(1 - \frac{\bar{p}_e \rho}{\bar{p}_e \bar{\rho}} \right) dy \right. \right. \right. \\ & \left. \left. - (\delta^* + \theta - \int_0^{\delta} \left(1 - \frac{\bar{p}_e \rho}{\bar{p}_e \bar{\rho}} \right) dy \right) \frac{d}{dx} (\ln \eta / \sigma) \right\} \left\{ \frac{1 + (\bar{L}_{ew} - 1)(1 - \bar{C}_{pw}^* \bar{T}_e / H_e)}{1 + (\bar{L}_{ew} - 1)(1 - \bar{C}_{pw}^* \bar{T}_e / H_e)} \right\} \frac{\bar{p}_{ew}}{\bar{p}_{2w}} \right. \\ & \left. - 1 \right] \left[C_H (H_r - H_w) + C_{H,D} \sum_i h_i (Y_{ie} - Y_{iw}) - H_e \phi \frac{d}{dx} (\ln \sigma) \right] \left[\frac{1}{H_e (\phi + \delta^* - \delta)} \right] dx \end{aligned} \quad (14.a)$$

Equation (14.a) gives α to within a constant in terms of $\sigma(x)$ and $\eta(x)$.

Applying a similar treatment to the species equation results in:

$$\begin{aligned} \bar{\beta}_i = & \frac{\sigma \lambda_i}{\xi} \left[\beta_i + \frac{d}{dx} (\ln \sigma) \int_0^{\delta} \psi \frac{\partial Y_i}{\partial y} dy - \frac{d}{dx} (\ln \lambda_i) \int_0^{\delta} \rho u Y_i dy \right. \\ & \left. - \int_0^{\delta} \dot{w}_i dy \right] + \eta \int_0^{\delta} \frac{\rho}{\bar{\rho}} \bar{w}_i dy \end{aligned} \quad (15)$$

At the wall, equation (15) becomes

$$\begin{aligned} \bar{\beta}_{i,w} = & \frac{\sigma \lambda_i}{\xi} \left[\beta_{i,w} + \rho_e u_e Y_{ie} \left\{ (\gamma_i + \delta^* - \delta) \frac{d}{dx} (\ln \lambda_i) + \gamma_i \frac{d}{dx} (\ln \sigma) \right\} \right. \\ & \left. - \int_0^{\delta} \dot{w}_i dy \right] + \eta \int_0^{\delta} \frac{\rho}{\bar{\rho}} \bar{w}_i dy \end{aligned} \quad (16.a)$$

where

$$\gamma_i = \int_0^{\delta} \frac{\rho u}{\rho_e u_e} \left(1 - \frac{Y_i}{Y_{ie}} \right) dy \quad (\text{specie thickness}) \quad (16.b)$$

From the definition of β_i (eq. 3.f) we can write at the wall

$$\frac{\bar{\beta}_{i,w}}{\beta_{i,w}} = \frac{\bar{\mu}_w \bar{p}_w}{\mu_w p_w} \frac{\lambda_i}{\eta} \frac{p_{ew} \bar{L}_{ew}}{\bar{p}_{ew} L_{ew}} \quad (17)$$

Combining equations (16.a) and (17) gives a third expression for $\xi(x)$

$$\begin{aligned} \xi(x) = \sigma \eta \frac{\bar{\mu}_w \bar{p}_w}{\mu_w p_w} \frac{\bar{p}_{ew} \bar{L}_{ew}}{p_{ew} L_{ew}} & \left[1 + \frac{p_e u_e Y_{ie}}{\beta_{i,w}} \left[\gamma_i \frac{d}{dx} (\ln \sigma) \right. \right. \\ & \left. \left. + (\gamma_i + \delta^* - \delta) \frac{d}{dx} (\ln \lambda_i) \right] - \int_0^\delta \frac{\bar{w}_i}{\beta_{i,w}} dy \right] \left[\frac{1}{1 - \frac{\eta}{\beta_{i,w}} \int_0^\delta \frac{\rho}{\bar{p}} \bar{w}_i dy} \right] \quad (18) \end{aligned}$$

Combining equations (9) and (18) gives an expression which $\alpha(x)$,

$\sigma(x)$, $\eta(x)$ and $\lambda_i(x)$ must satisfy.

$$\begin{aligned} & 1 + \frac{p_e u_e Y_{ie}}{\beta_{i,e}} \left[\gamma_i \frac{d}{dx} (\ln \sigma) + (\gamma_i + \delta^* - \delta) \frac{d}{dx} (\ln \lambda_i) \right] - \frac{\int_0^\delta \bar{w}_i dy}{\beta_{i,w}} \\ & = \left[1 - \frac{\eta}{\beta_{i,w}} \int_0^\delta \frac{\rho}{\bar{p}} \bar{w}_i dy \right] \left[\frac{1 + (L_{ew} - 1)(1 - c_{pw}^* T_e / H_e)}{1 + (\bar{L}_{ew} - 1)(1 - \bar{c}_{pw}^* \bar{T}_e / \bar{H}_e)} \right] \frac{\bar{L}_{ew}}{L_{ew}} \left[1 + \right. \\ & \quad \left. \frac{H[\phi \frac{d}{dx} (\ln \sigma) + (\phi + \delta^* - \delta) \frac{d}{dx} (\ln \alpha)]}{C_H (H_r - H_w) + C_{HD} \sum_i \lambda_i (Y_{ie} - Y_{iw})} \right] \quad (19) \end{aligned}$$

Note that equation (19) still involves $\eta(x)$, even if the low speed flow is frozen ($\bar{w}_i = 0$) or in equilibrium, since $\ln \alpha$ is a function of η . (See equation 14.a) Solving equation (19) we get λ_i to within a constant in terms of σ and η since α is given in terms of σ and η by equation (14.a).

$$\begin{aligned}
\ln \lambda_i = & \int_0^x \left[\frac{\bar{L}_{ew}}{\bar{L}_{ew}} \left[1 - \frac{\eta}{\bar{\beta}_{iw}} \int_0^{\delta} \frac{\rho}{\bar{\rho}} \bar{w}_i dy \right] \left[\frac{1 + (\bar{L}_{ew} - 1)(1 - \bar{c}_{pw}^* T_e / \bar{H}_e)}{1 + (\bar{L}_{ew} - 1)(1 - \bar{c}_{pw}^* \bar{T}_e / \bar{H}_e)} \right] \right. \\
& \left. \left[1 + \frac{H \left[\phi \frac{d}{dx} (\ln \sigma) + (\phi + \delta^* - \delta) \frac{d}{dx} (\ln \alpha) \right]}{C_H (H_r - H_w) + C_{HD} \sum_i h_i (Y_{ie} - Y_{iw})} \right] + \frac{1}{\bar{\beta}_{iw}} \int_0^{\delta} \bar{w}_i dy - 1 \right) \\
& \left. \left[\frac{\beta_{iw}}{\rho_e u_e Y_{ie} (\delta_i + \delta^* - \delta)} \right] - \frac{\delta_i \frac{d}{dx} (\ln \sigma)}{\delta_i + \delta^* - \delta} \right\} dx \quad (19.a)
\end{aligned}$$

Now if we assume that zero pressure gradients transform to zero pressure gradients (as was assumed by Coles and shown to be reasonable by Crocco), from equation (7) we have that $\ln \eta / \sigma = \text{constant}$, as was done in reference 4.

Now the last variable $\mathcal{O}(x)$ must be determined in some experimental way since there are no more applicable conditions available. Rosenbaum⁽⁴⁾ evaluated \mathcal{O} by relating it to Reynolds number invariance at the edge of the laminar sub-layer which separates the viscous from the non-viscous boundary layer. (See reference 4, pp 15 ff.)

Now since $\mathcal{O}(x)$ is known the other transformation variables are all known to within a constant.

Since the transformation variables are now known it is possible to see how the Prandtl and Lewis numbers transform.

In the same manner as Crocco, we can write, after some algebra, that from the momentum and energy equations, using equations (8) and (10).

$$\frac{\alpha \frac{\partial H}{\partial y} + (\bar{P}_e - 1) \frac{\sigma^2}{\eta^2} u \frac{\partial u}{\partial y} + (\bar{L}_e - 1) \left[\alpha \frac{\partial H}{\partial y} - \frac{\sigma^2}{\eta^2} u \frac{\partial u}{\partial y} - \alpha \bar{C}_p^* \bar{T}_e / \bar{H}_e \frac{\partial H}{\partial y} \right]}{\alpha \bar{P}_e} =$$

$$\frac{\tau \left\{ \frac{\partial H}{\partial y} + (\bar{P}_e - 1) u \frac{\partial u}{\partial y} + (\bar{L}_e - 1) \left[\frac{\partial H}{\partial y} - u \frac{\partial u}{\partial y} - \bar{C}_p^* \bar{T}_e / \bar{H}_e \frac{\partial H}{\partial y} \right] \right\} + \bar{P}_e \frac{\partial u}{\partial y} \left[\frac{d}{dx} (\ln \sigma) \int_y^{\delta} \psi \frac{\partial H}{\partial y} dy - \frac{d}{dx} (\ln \eta) \int_y^{\delta} \rho u H dy \right]}{\bar{P}_e \left\{ \tau + \frac{d}{dx} (\ln \sigma) \int_y^{\delta} \psi \frac{\partial u}{\partial y} dy + \frac{dP}{dx} \int_y^{\delta} \left(1 - \frac{\bar{P}_e \rho}{\bar{P} \bar{P}_e} \right) dy + \bar{P}_e u_e^2 \frac{d}{dx} (\ln \eta / \sigma) \int_y^{\delta} \frac{\rho}{\bar{P}_e} \left(\frac{u^2}{u_e^2} - \frac{\bar{P}_e}{\bar{P}} \right) dy \right\}} \quad (20)$$

Equation (20) reduces to Crocco's equation 3.13 for zero pressure gradient

$$\frac{dP}{dx} = \frac{du_e}{dx} = 0, \quad \alpha = \text{constant}, \quad L_e = \bar{L}_e = 1, \quad \text{and}$$

$\sigma/\eta = \text{constant}$. Note that the Prandtl and Lewis numbers are coupled.

We get a second relation in the same manner from the Species and Momentum equations, using the definitions of β_i and τ , and replacing the Schmidt numbers by Prandtl over Lewis numbers.

$$\frac{\bar{L}_e}{\bar{P}_e} = \left[\tau L_e / \bar{P}_e + \left\{ \frac{d}{dx} (\ln \sigma) \int_y^{\delta} \psi \frac{\partial Y_i}{\partial y} dy - \frac{d}{dx} (\ln \lambda_i) \int_y^{\delta} \rho u Y_i dy - \int_y^{\delta} \bar{\omega}_i dy + \frac{\eta \bar{L}_e}{\sigma \lambda_i} \int_y^{\delta} \frac{\rho}{\bar{P}} \bar{\omega}_i dy \right\} \frac{\partial u}{\partial y} \right] / \left[\tau + \frac{d}{dx} (\ln \sigma) \int_y^{\delta} \psi \frac{\partial u}{\partial y} dy + \frac{dP}{dx} \int_y^{\delta} \left(1 - \frac{\bar{P}_e \rho}{\bar{P} \bar{P}_e} \right) dy + \bar{P}_e u_e^2 \frac{d}{dx} (\ln \eta / \sigma) \int_y^{\delta} \frac{\rho}{\bar{P}_e} \left(\frac{u^2}{u_e^2} - \frac{\bar{P}_e}{\bar{P}} \right) dy \right] \quad (21)$$

In general we can solve equations (20) and (21) for $L_e(\bar{L}_e, \bar{P}_e)$ and $\bar{P}_e(\bar{L}_e, \bar{P}_e)$, and consequently the Lewis and Prandtl numbers are not independent. To illustrate, a very simple example will be shown.

Assume:

- 1) that zero pressure gradients transform to zero pressure gradients.

(Implies $\sigma/\eta = \text{constant}$)

2) that the pressure gradient is zero.

(Implies $P = \text{constant}$, $u_e = \text{constant}$, $\rho_e = \text{constant}$)

3) that $\sigma(x) = \text{constant}$, σ^* .

(Implies streamlines are proportional and that $\eta = \text{constant}$,

η^* from 1.)

4) $\dot{w}_i = \bar{\dot{w}}_i = 0$ either equilibrium or frozen flow in each regime.

In order that the 3 equations (9), (13) and (18) be consistent, we have

$$\xi = \sigma^* \eta^* \frac{\rho_w \mu_w}{\bar{\rho}_w \bar{\mu}_w} \quad (22.a)$$

$$\alpha = \text{constant} = \alpha^* \quad (22.b)$$

$$\lambda_i = \text{constant} = \lambda_i^* \quad (22.c)$$

$$\frac{\bar{P}_{ew} H_e + \bar{P}_{ew} (L_{ew} - 1)(H_e - C_{pw}^* T_e)}{H_e} = \frac{\bar{P}_{ew} \bar{H}_e + \bar{P}_{ew} (\bar{L}_{ew} - 1)(\bar{H}_e - \bar{C}_{pw}^* \bar{T}_e)}{\bar{H}_e} \quad (22.e)$$

$$\frac{\bar{P}_{ew}}{\bar{P}_{ew}} = \frac{\bar{L}_{ew}}{\bar{L}_{ew}} \quad (22.f)$$

Note that in this simple case, equation (22.e) gives an extra condition to be satisfied by the Lewis and Prandtl number transformation, thus enabling the solution of 3 of the numbers in terms of a single quantity,

since (22.f) will be shown to be exactly the same as equation (21).

Using the above relations equation (21) becomes (Note that it agrees identically with equation (22.f)).

$$\frac{\bar{L}_e}{\bar{P}_e} = \frac{L_e}{P_e} \quad (23.a)$$

and equation (20) becomes

$$P_e \left\{ \frac{\partial H}{\partial y} + \frac{\sigma^{*2}}{\alpha^* \eta^{*2}} (\bar{P}_e - 1) u \frac{\partial u}{\partial y} + (\bar{L}_e - 1) \left[\frac{\partial H}{\partial y} - \frac{\sigma^{*2}}{\alpha^* \eta^{*2}} u \frac{\partial u}{\partial y} - \frac{\bar{C}_p^* \bar{T}_e}{\bar{H}_e} \frac{\partial H}{\partial y} \right] \right\} =$$

$$\bar{P}_e \left\{ \frac{\partial H}{\partial y} + (P_e - 1) u \frac{\partial u}{\partial y} + (L_e - 1) \left[\frac{\partial H}{\partial y} - u \frac{\partial u}{\partial y} - C_p^* \frac{T_e}{H_e} \frac{\partial H}{\partial y} \right] \right\} \quad (23.b)$$

Now assuming $L_e = \text{constant} = L_{ew}$

$$\bar{L}_e = \text{constant} = \bar{L}_{ew}$$

$$P_e = \text{constant} = P_{ew}$$

$$\bar{P}_e = \text{constant} = \bar{P}_{ew}$$

$$\bar{C}_p^* = \text{constant} = \bar{C}_{pw}^*$$

$$C_p^* = \text{constant} = C_{pw}^*$$

(23.c)

equation (23.b) can be integrated from $y = 0$ to ∞ giving

$$P_{ew} \left\{ H_e - H_w + \frac{\sigma^{*2}}{\alpha^* \eta^{*2}} (\bar{P}_{ew} - 1) \frac{u_e^2}{2} + (\bar{L}_{ew} - 1) \left[H_e - H_w - \frac{\sigma^{*2}}{\alpha^* \eta^{*2}} \frac{u_e^2}{2} \right. \right. \quad (24)$$

$$\left. - \frac{\bar{C}_{pw}^* \bar{T}_e}{\bar{H}_e} (H_e - H_w) \right\} = \bar{P}_{ew} \left\{ H_e - H_w + (P_{ew} - 1) \frac{u_e^2}{2} + (L_{ew} - 1) \left[H_e - H_w - \frac{u_e^2}{2} - C_{pw}^* \frac{T_e}{H_e} (H_e - H_w) \right] \right\}$$

Also using relations (23.c) in (22.e) and a little algebra we get

$$L_{ew} = \frac{1 \pm \left[1 + \frac{4 C_{Pw}^* T_e \bar{H}_e}{\bar{C}_{Pw}^* \bar{T}_e \bar{H}_e} L_{ew} (\bar{L}_{ew} - 1) \right]^{1/2}}{2} \quad (25)$$

From physical considerations we see that the plus sign must be used.

Now using equation (23.a) in equation (24) and solving for \bar{P}_{ew} , we have after some algebra

$$\bar{P}_{ew} = \frac{\frac{H_e - H_w}{\frac{1}{2} U_e^2} \left\{ (1 - \bar{L}_{ew}) \frac{\bar{C}_{Pw}^* \bar{T}_e}{\bar{H}_e} - \frac{\bar{L}_{ew} (1 - L_{ew}) C_{Pw}^* T_e}{L_{ew} \bar{H}_e} \right\} + \bar{L}_{ew} \left(1 - \frac{\sigma^{*2}}{\alpha^{*2} \eta^{*2}} \right)}{\left(1 - \frac{\sigma^{*2}}{\alpha^{*2} \eta^{*2}} \right)} \quad (26)$$

Thus we have arrived at the equations (26), (25), (23.a) describing the transformation of Prandtl and Lewis numbers in the two flow regimes, if we know the values of σ^* , α^* , and η^* . Note that once one of the numbers in either flow is known the rest are automatically known by equations (26), (25), and (23.a).

Since available experimental data shows rather wide variations (from unity) in the Lewis and Prandtl numbers applicable to high speed mixing layers, we must conclude that the Coles-Crocco transformation scheme is of very dubious applicability to reacting flows. The correspondence between Lewis and Prandtl numbers imposed by the mathematical transformation will usually not be satisfied physically, except in the special case where transport of each species, momentum and energy occur at equal rates.

Figure 1 shows how the numbers vary for a given \bar{T}_{ew} for the following conditions

	high velocity flow	low velocity flow
U_e	15,000 ft/sec	1000 ft/sec
T_e	298 °K	298 °K
T_w	894 °K	894 °K
H_e	$1.075 \times 10^{11} \text{ cm}^2/\text{sec}^2$	$3.47 \times 10^9 \text{ cm}^2/\text{sec}^2$
H_w	$9.03 \times 10^9 \text{ cm}^2/\text{sec}^2$	$9.03 \times 10^9 \text{ cm}^2/\text{sec}^2$
C_{pw}^*	7.0 cal/gr-mole	7.0 cal/gr-mole

The values of σ^* , η^* , α^* used were the following

$$\left[\frac{\sigma^*}{\eta^*} \right]^2 = \left[\frac{\bar{U}_e}{U_e} \right]^2 = .0044 \quad (\text{from equation 6})$$

$$\alpha^* = \bar{H}_e / H_e = .0323 \quad (\text{from reference 2})$$

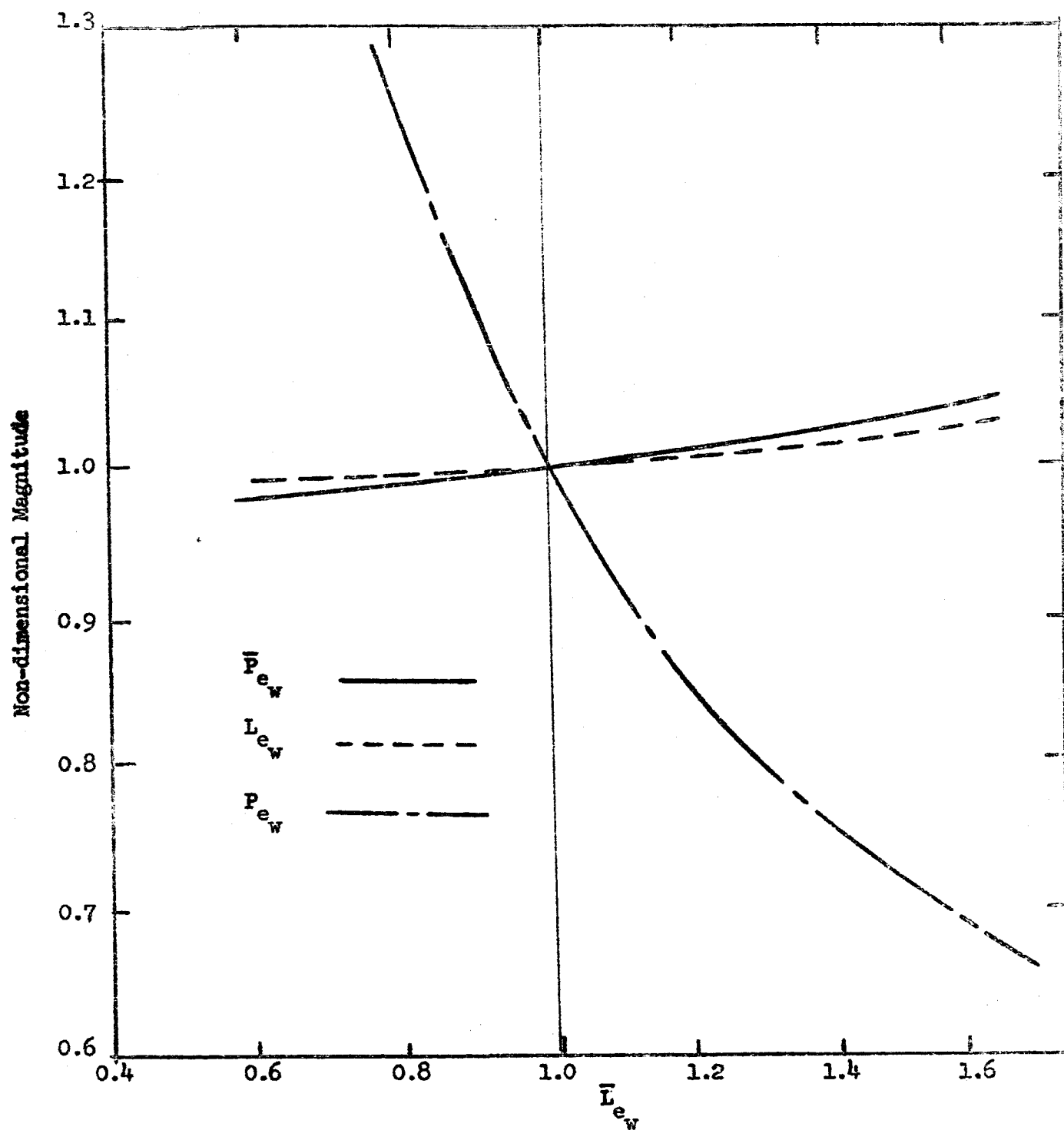


Figure 1. Transformation of the Prandtl and Lewis Numbers

4. Future Work

The chemical equilibrium solutions for the exhaust plume - air free shear layer (which have been provided by NASA) will be used in our previously developed calculation scheme to determine the relative importance of various reactions. The results will be compared with those of our previously accomplished study of near-frozen mixing profiles to determine the nature of the important reactions and thus permit deduction of non-equilibrium scaling laws. The rate of macroscopic mixing of turbulent eddies will be calculated for the plume - air mixing layer. These mixing rates will be compared with the chemical reaction rates to attempt to elucidate the role of macroscopic turbulent mixing in ignition delay in the afterburning region.

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